On Monotonic Solutions of Nonlinear Quadratic Integral Equation of Convolution Type

Author's Details:
Wagdy G. El-Sayed[1], Mahmoud M. El-Borai[2], Mohamed M.A. Metwali[3],
Nagwa I. Shemais[4] [1,2] Department of mathematics and computer science, Faculty of Science, Alexandria University, Alexandria, Egypt [3,4] Department of mathematics, Faculty of Science , Damanhour University, Damanhour, Egypt
[1]wagdygoma@alexu.edu.eg, [2]m m elborai@yahoo.com,
[3]m.metwali@yahoo.com,[4] n.ibrahim90@yahoo.com

Abstract:
Using the technique of a suitable measure of non-compactness and the Darbo fixed point, we investigate the existence of nondecrasing solutions for a quadratic nonlinear -integral equation of convolution type. Our investigations take place in $C(\mathbb{R}_+)$, *the Banach space of real and continuous functions defined on* $\mathbb{R}_+$, $\mathbb{R}_+ := [0,\infty)$. *An example is also discussed to indicate the natural realizations of our abstract result.*
Keywords: *Quadratic integral equation, measure of noncompactnes, existence and uniqueness, Darbo's fixed point theorem.*

1. Introduction.

It is well Known that Several problems in many branches in real world such as in physics, engineering,...ect, can be modeled as Quadratic integral equations see for example([4], [6], [7], [17]), and the references therein, which are often applicable in the kinetic theory of gases, in the traffic theory, and in the radiative transfer. Especially the so called quadratic integral equation of Chandrsekher type can be seen in many application (cf.([9],[10],[30],[12],[29])).

In [33], the authors studied the nondecreasing solutions of a Quadratic integral equation of urysohn type

$$x(t) = a(t) + + f\left(t,\, x(t)\right)\int_0^1 u\left(t,\, s,\, x(s)\right)\, ds, t \in I = [0,1].$$

the authors in[18] studied the existence of integral solutions of the following integral equation

$$x(t) = f_1(t, \int_0^t k(t,\ s)\ f_2(s, x(s))ds).$$

In[19], the authors studied the existence of solutions for the perturbed functional integral equations of convolution type

$$x(t) = f_1\left(t,\, x(t)\right) + f_2\left(t,\ \int_0^\infty k(t-s)\ Q(x)(s)\ ds\right), t \in \mathbb{R}_+$$

the space $L^p(\mathbb{R}_+)$ in (the space of lebesgue integrable functions on $\mathbb{R}_+$).

In [1], the authors studied the solvability of the following nonlinear quadratic integral equation

$$x(t) = g(t) + (Tx)(t)\int_0^t k(t,s)\, f\left(s, x\left(\varphi(s)\right)\right)\, ds, t \in [0, M].$$

2. Notation And Auxiliary Results.

We need to recall some basic concepts that will need later. Let E be an infinite dimensional Banach space with norm $\|.\|$ and zero element θ. Denote by $B_r = B(\theta, r)$ the closed ball centered at θ and with radius r. If X is nonempty and subset of E, we denote the symbols $\bar{X}$ and $ConvX$ stand for the closure and closed convex hull of X, respectively. Moreover, we use the standard notation $X+Y$ and λX for algebraic operations on sets.

Further on, we denote by M_E the family of all nonempty and bounded subsets of E and N_E the family of all nonempty relatively compact sets.

Definition2.1. (Measure of noncompactness)[11]

A mapping $\mu : M_E \to [0,\infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions,

1^0 the family $ker\mu = \{X \in M_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset N_E$, where $\ker\mu$ is called the kernel of the measure μ.

2^0 $X \subset Y \Rightarrow \mu(X) \le \mu(y)$,

3^0 $\mu(ConvX) = \mu(X) = \mu(\bar{X})$,

4^0 $\mu[\lambda X + (1-\lambda)Y] \le \lambda\mu(X) + (1-\lambda)\mu(Y), \lambda \in [0,1]$,

5^0 If $= X_n \in M_E, X_n = \bar{X}_n$ and $X_{n+1} \subset X_n$ for $n = 1,2,...$ and if

$\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \varphi$(nonempty and compact).

The family $\ker\mu$ described above is called the kernel of the measure of noncompactness μ. For further details concerning of measures of noncompactness and its properties, we refer to [11]. In what follows we will work in the Banach space $C(\mathbb{R}_+)$ consisting of all real functions defined and continuous on $\mathbb{R}_+ = [0,\infty)$. The space $C(\mathbb{R}_+)$ is equipped with the standard norm $\|x\| = \max\{|x(t)| : t \in \mathbb{R}_+\}$. Now, let us recall the definition of a measure of noncompactness in $C(\mathbb{R}_+)$ which will be used in our investigations. This measure was introduced in [14]. Let us fix a nonempty and bounded subset X of $C(\mathbb{R}_+)$ For $x \in X$ and $\varepsilon > 0$. we let $w(x,\varepsilon) = \sup\{|x(t) - x(s)| : t,s \in \mathbb{R}_+, |t-s| \le \varepsilon\}$,

$$w(X,\varepsilon) = \sup\{w(x,\varepsilon) : x \in X\},$$

And
$$w_0(X) = \lim_{\varepsilon\to 0} w(X, \varepsilon).$$

Also, let us define the following quantities,

$$d(x) = sup\{ |x(s) - x(t)| - [x(s) - x(t)] t,s \in \mathbb{R}_+, \sim\sim t \le s\},$$

$$d(X) = sup\{d(x) : x \in X\}.$$

Notice that $d(X) = 0$ if and only if all functions belonging to X are nondecreasing on $\mathbb{R}_+$. It can proved in [14], that the function $\mu : M_{C(\mathbb{R}_+)} \to \mathbb{R}$ given by

$\mu(X) = w_0(X) + d(X)$,

is measure of noncompactness on $C(\mathbb{R}_+)$. The kernel $\ker\mu$ of this measure contains nonempty and bounded sets X such that functions from X are equicontinuous and nondecreasing on $\mathbb{R}_+$.

In the end of this section, we recall the fixed point of Darbo which enables us to prove the solvability of several integral equations considered in nonlinear analysis. To quote this theorem we need the following definitions.

Definition2.2. [11] The function $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheodory condition if it satisfies the following two conditions:

(1) f is measurable in $t \in I$ for any $x \in \mathbb{R}$.

(2) f is continuous in $x \in \mathbb{R}$ for almost all $t \in I$.

Definition2.3. (Darbo condition) [15]
Let Ω be a nonempty subset of a Banach space E and let $A:\Omega \to E$ be a continuous operator which transforms bounded sets onto bounded ones. We say that A satisfies the Darbo condition (with a constant $k \geq 0$) with respect to a measure of noncompactness μ if for any bounded subset X of Ω, we have $\mu(AX) \leq k\mu(X)$.
Note that, if A satisfies the Darbo condition with $k < 1$, then it is called a contraction operator with respect to μ.
Theorem2.1. (Darbo fixed point theorem) [5]
Let Ω be a nonempty, bounded, closed and convex subset of E and let $f:\Omega \to \Omega$ be a continuous transformation which is a contraction with respect to the measure of noncompactness μ, i.e. there exists a constant $k \in [0,1)$ such that

$$\mu(fX) \leq k\mu(X),$$

for any nonempty subset X of μ. Then f has at least one fixed point in the set μ.
Remark2.1. Under the assumptions of the above theorem, it can be shown that the set fix F of fixed points of Fbelonging to μ is a member of the family ker μ (cf., [11]). This fact permits us to characterize solutions of considered operator equations.

3. Main result

In this section, we will study the nonlinear quadratic integral equation of convolution type having the form,

$$x(t) = g(t) + h(t,\, x(t)) \int_0^\infty k(t-s)\, f(s,\, x(s))\, ds, \mathrm{t} \in \mathbb{R}_+. \qquad (1)$$

All functions appearing in equation (1) are known while $x = x(t)$ is an unknown function. For further purposes let us also recall that the function $h = h(t,\, x)$ involved in (1) generates the operator H defined by the formula,

$$(H\,x)(t) = h(t,\, x(t)), \qquad (2)$$

where $x = x(t)$ is an arbitrary function defined on $\mathbb{R}$ Such as an operator is called the superposition operator and have several interesting properties([11]). In what follows, we will investigate the integral equation (1) assuming the following conditions are satisfied.

(i) $g \in C(\mathbb{R}_+)$ is continuous nondecreasing and nonnegative.

(ii) $h:\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous function such that there exists a constant $a \geq 0$ such that

$$|\ h(t,\, x) - h(t,\, y)| \leq a|\ x\ -\ y|$$

for all $t \in \mathbb{R}_+$ and $x, y\ \in \mathbb{R}$.Moreover, $h:\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$.

(iii) $d(Hx) \leq ad(x)$ for any nonnegative function $x \in C(\mathbb{R}_+)$, where H is the superposition operator defined in (2).

(iv) $k:\mathbb{R}_+ \to \mathbb{R}$ is continuous and nondecreasing where

$$K = \sup\{\int_0^\infty k(t-s)ds : t,s \in \mathbb{R}_+\}.$$

(v) f $:\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous function and nondecreasing such that there exists a nondecreasing function $m:\mathbb{R}_+ \to \mathbb{R}_+$, such that $|f(s,x)| \leq m(|x|)$ for all $s \in \mathbb{R}_+$ and $x \in \mathbb{R}$.

(vi) There exist $r_0 > 0$ with

$$\|g\| + (ar_0 + b)Km(r_0) \leq r_{0,}$$

such that $aKm(r_0)<1$, where $b=max\{h(t,0):t\in\mathbb{R}_+\}$.

Now we are in position to present our main result.

Theorem 3.1. Under the assumptions (i)-(vi), equation (3.1) has at least one solution $x=x(t)$ which belong the space $C(\mathbb{R}_+)$ and nondecreasing.

Proof: Let us consider the operator U defined on the space $C(\mathbb{R}_+)$ in the following way,

$$(Ux)(t)=g(t)+h(t,x(t))\int_0^\infty k(t-\tau)f(\tau,x(\tau))d\tau,$$

taking into account assumptions (i), (ii), (iv) and the properties of the superposition operator, we infer that the function Ux is continuous on $\mathbb{R}_+$ for any function $x\in C$ (the operator U transforms the space $C(\mathbb{R}_+)$ into itself).

$$\begin{aligned}|(Ux)(t)| &\le |g(t)|+|h(t,x(t))|\int_0^\infty |k(t-\tau)||f(\tau,x(\tau))|d\tau\\
&\le |g(t)|+\left[|h(t,x(t))-h(t,0)|+|h(t,0)|\right]\int_0^\infty |k(t-\tau)|m(|x(\tau)|)d\tau\\
&\le |g(t)|+\left(a|x(t)|+|h(t,0)|\right)\int_0^\infty |k(t-\tau)|m(|x(\tau)|)d\tau\\
&\le |g(t)|+\left(a|x(t)|+h(t,0)\right)m(\|x\|)\int_0^\infty |k(t-\tau)|d\tau\\
&\le |g(t)|+\left(a|x(t)|+h(t,0)\right)m(\|x\|)K.\end{aligned}$$

The above inequality yields

$$\|Ux\|\le\|g\|+(a\|x\|+b)Km(\|x\|),$$

where $b=\max\{h(t,0):t\in\mathbb{R}_+\}$.

Hence, keeping in mind assumption (vi), we deduce that there exists $r_0>0$ and such that the operator U on the subset $B_{r_0}^+$ of the ball B_{r_0} defined as follows,

$$B_{r_0}^+=\{x\in B_{r_0}:x(t)>0\}.$$

Obviously, the set $B_{r_0}^+$ is nonempty, bounded, closed, and convex. Hence in view of our assumptions we infer that U transforms the ball $B_{r_0}^+$ into itself. Then, we show that U is continuous on the set $B_{r_0}^+$. To do this, let us fix $\varepsilon\ge 0$ and take arbitrarily $x,y\in B_{r_0}^+$ such that $\|x-y\|\le\varepsilon$, then we have for $t\in\mathbb{R}_+$,

$$\begin{aligned}&|(Ux)(t)-(Uy)(t)|\\
&=\left|h(t,x(t))\int_0^\infty k(t-\tau)f(\tau,x(\tau))d\tau-h(t,y(t))\int_0^\infty k(t-\tau)f(\tau,y(\tau))d\tau\right|\\
&\le\left|h(t,x(t))\int_0^\infty k(t-\tau)f(\tau,x(\tau))d\tau-h(t,x(t))\int_0^\infty k(t-\tau)f(\tau,y(\tau))d\tau\right|\\
&+\left|h(t,x(t))\int_0^\infty k(t-\tau)f(\tau,y(\tau))d\tau-h(t,y(t))\int_0^\infty k(t-\tau)f(\tau,y(\tau))d\tau\right|\\
&\le[|h(t,x(t))-h(t,0)|+|h(t,0)|]\int_0^\infty k(t-\tau)|f(\tau,x(\tau))-f(\tau,y(\tau))|d\tau\\
&+|h(t,x(t))-h(t,y(t))|\int_0^\infty |k(t-\tau)f(\tau,y(\tau))|d\tau\\
&\le\left(a|x(t)|+b\right)\int_0^\infty k(t-\tau)|f(\tau,x(\tau))-f(\tau,y(\tau))|d\tau+a|x(t)-y(t)|\end{aligned}$$

$$\times\int_0^\infty |k(t-\tau)|\, m(|y(\tau)|)d\tau$$
$$\leq (ar_0+b)K\beta_{r_0}(\varepsilon)+a\varepsilon Km(r_0),$$

where we denoted

$$\beta_{r_0}(\varepsilon)=\sup\{|f(\tau,x(\tau))-f(\tau,y(\tau))|:\tau\in\mathbb{R}_+, x,y\in[0,r_0], |x-y|\leq\varepsilon\}.$$

Observe that $\beta_{r_0}(\varepsilon)\to 0$ as $\varepsilon\to 0$ which is a consequence of the uniform continuity of the function f on the set $\mathbb{R}_+\times[0,r_0]$. Next, we get

$$\|Ux-Uy\|\leq (ar_0+b)K\beta_{r_0}(\varepsilon)+a\varepsilon Km(r_0),$$

which yields the continuity of the operator U on the set $B_{r_0}^+$. Now, let us take a nonempty set $x, X\in B_{r_0}^+$. Further fix arbitrary $\varepsilon>0$ and choose $x\in X$ and $t,s\in\mathbb{R}_+$, such that $|x-y|\leq\varepsilon$. Then, taking into account our assumption, we obtain

$$|(Ux)(s)-(Ux)(t)|\leq|g(s)-g(t)|$$
$$+\left|h(s,x(s))\int_0^\infty k(s-\tau)f(\tau,x(\tau))d\tau-h(t,x(t))\int_0^\infty k(t-\tau)f(\tau,x(\tau))d\tau\right|$$
$$\leq w(g,\varepsilon)+\left|h(s,x(s))\int_0^\infty k(s-\tau)f(\tau,x(\tau))d\tau-h(t,x(t))\int_0^\infty k(s-\tau)f(\tau,x(\tau))d\tau\right|$$
$$+\left|h(t,x(t))\int_0^\infty k(s-\tau)f(\tau,x(\tau))d\tau-h(t,x(t))\int_0^\infty k(t-\tau)f(\tau,x(\tau))d\tau\right|$$
$$\leq w(g,\varepsilon)+|h(s,x(s))-h(t,x(t))|\int_0^\infty k(s-\tau)|f(\tau,x(\tau))|d\tau$$
$$+|h(t,x(t))|\int_0^\infty |k(s-\tau)k(t-\tau)||f(\tau,x(\tau))|d\tau$$
$$\leq w(g,\varepsilon)+[|h(s,x(s))-h(t,x(s))|+|h(t,x(s))-h(t,x(t))|]$$
$$\times\int_0^\infty k(s-\tau)m(|x(\tau)|)d\tau$$
$$+[|h(t,x(t))-h(t,0)|+|h(t,0)|]\int_0^\infty |k(s-\tau)-k(t-\tau)|m(|x(\tau)|)d\tau$$
$$\leq w(g,\varepsilon)+[w_{r_0}(h,\varepsilon)+a|x(s)-x(t)|]m(\|x\|)$$
$$\times\int_0^\infty |k(s-\tau)|d\tau+(a\|x\|+b)m(\|x\|)\int_0^\infty |k(s-\tau)-k(t-\tau)|d\tau$$
$$\leq w(g,\varepsilon)+[w_{r_0}(h,\varepsilon)+aw(x,\varepsilon)]Km(r_0),$$

where we have denoted

$$w_{r_0}(h,\varepsilon)=\sup\{h(t,x)-h(s,x):t,s\in\mathbb{R}_+, |t-s|\leq\varepsilon, x\in[0,r_0]\}.$$

Notes that in view of the uniform continuity of the function h on the set $\mathbb{R}_+\times[0,r_0]$, we have that $w_{r_0}(h,\varepsilon)\to 0$ as $\varepsilon\to 0$. This fact with the above estimate allow us to have,

$$w_0(UX)\leq aKm(r_0)w_0(X). \qquad (3)$$

Further, fix arbitrary $x\in X$ and $t,s\in\mathbb{R}_+$ with $t\leq s$. Then using the assumptions in our theorem, we get

$$|(Ux)(s)-(Ux)(t)|-[(Ux)(s)-(Ux)(t)]$$
$$=\left|g(s)-g(t)+h(s,x(s))\int_0^\infty k(s-\tau)f(\tau,x(\tau))d\tau-h(t,x(t))\int_0^\infty k(t-\tau)f(\tau,x(\tau))d\tau\right|$$
$$-[g(s)-g(t)+h(s,x(s))\int_0^\infty k(s-\tau)f(\tau,x(\tau))d\tau$$
$$-h(t,x(t))\int_0^\infty k(t-\tau)f(\tau,x(\tau))d\tau]$$
$$\leq\{|g(s)-g(t)|-[g(s)-g(t)]\}$$
$$=\left|h(s,x(s))\int_0^\infty k(s-\tau)f(\tau,x(\tau))d\tau-h(t,x(t))\int_0^\infty k(t-\tau)f(\tau,x(\tau))d\tau\right|$$

$-[h(s,x(s))\int_0^\infty k(s-\tau)f(\tau,x(\tau))d\tau - h(t,x(t))\int_0^\infty k(t-\tau)f(\tau,x(\tau))d\tau]$

$\le \left|h(s,x(s))\int_0^\infty k(s-\tau)f(\tau,x(\tau))d\tau - h(t,x(t))\int_0^\infty k(s-\tau)f(\tau,x(\tau))d\tau\right|$

$+\left|h(t,x(t))\int_0^\infty k(s-\tau)f(\tau,x(\tau))d\tau - h(t,x(t))\int_0^\infty k(t-\tau)f(\tau,x(\tau))d\tau\right|$

$-[h(s,x(s))\int_0^\infty k(s-\tau)f(\tau,x(\tau))d\tau - h(t,x(t))\int_0^\infty k(s-\tau)f(\tau,x(\tau))d\tau]$

$-[h(t,x(t))\int_0^\infty k(s-\tau)f(\tau,x(\tau))d\tau - h(t,x(t))\int_0^\infty k(t-\tau)f(\tau,x(\tau))d\tau]$

$\le |h(s,x(s))-h(t,x(t))|\int_0^\infty |k(s-\tau)|m(|x(\tau)|)d\tau$

$+|h(t,x(t))|\int_0^\infty |k(s-\tau)-k(t-\tau)|m(|x(\tau)|)d\tau$

$-[h(s,x(s))-h(t,x(t))]\int_0^\infty k(s-\tau)|f(\tau,x(\tau))|d\tau$

$-h(t,x(t))\int_0^\infty [k(s-\tau)-k(t-\tau)]f(\tau,x(\tau))d\tau$

$\le (|h(s,x(s))-h(t,x(t))|-[h(s,x(s))-h(t,x(t))])$

$\times\int_0^\infty k(s-\tau)|f(\tau,x(\tau))|d\tau$

$\le d(Hx)\int_0^\infty k(s-\tau)f(\tau,x(\tau))d\tau$

$\le d(Hx)Km(r_0) \le aKm(r_0)d(x).$

Then, we have

$$d(UX) \le aK\, m(r_0)\, d(X). \tag{4}$$

Now from equation (3), (4), and the definition of measure of noncompactness given in section 2, we get

$$\mu(UX) \le aK\, m(r_0)\, \mu(X).$$

Hence, taking into account that $aK\, m(r_0)<1$ and the theorem 2.1we obtain that the integral equation (1) has at least one solution in $C(\mathbb{R}_+)$. Moreover, in view of Remark 2.1 and the description of the kernel of a measure of noncompactness μ, we deduce that all solutions of the integral equation (1) belonging to the set $B_{r_0}^+$ are nondecreasing on $\mathbb{R}_+$. This complete the proof.

Corollary 3.1. Observe that assuming that $g(t)>0$, for $t\in\mathbb{R}_+$, we infer that all solutions of the integral equation (1) belonging to the set $B_{r_0}^+$ are continuous, nondecreasing, and positive on $\mathbb{R}_+$.

proof

In the case when the assumptions of theorem are satisfied, then the operator U has a fixed point say X i.e

$$UX = X,$$

then

$$\mu(X)=\mu(U\,X)\le a\,K\,m(r_0)\,\mu(X),$$

hence

$$(1-aKm(r_0))\,\mu(\mathrm{X})\le 0.$$

Since $a\,K\,m(r_0)< 1$ then

$$0 < (1-\, aK\, m(r_0) < 1,$$

then

$$\mu(X)=0.$$

So

$$\omega(X) + d(X) = 0,$$

then $\omega(X) = d(X) = 0$, then the solutions of the integral equation (1) will be monotonic nondecreasing. We will study the uniqueness of solution for equation (1).

4. Existence of a unique solution

Theorem 4.1. Let the assumptions of the above Theorem be satisfied but the assumptions (v)(vi), will replaced by
the following assumptions

(a) $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ Satisfies lipschitz condition in x with constant $l \geq 0$, such that

$$\left| f\left(t, x(t)\right) - f\left(t, y(t)\right) \right| \leq l|x - y|,$$

also assume that $|f(t, x(t))| \leq l_1 |x|$, l_1 is positive constant

(b) $(ar_0 + b)Kl + aKl_1 r_0 < 1.$

Proof:

From assumption (ii), we have

$|h(t, x(t))| \leq |h(t, x(t)) - h(t, 0)| + |h(t, 0)| \leq a|x| + b \Rightarrow |h(t, x(t))| \leq a|x| + b,$

for the uniqueness of solution of equation(3.1), let $x(t)$, $y(t)$ be any two solutions for the equation (1), in B_{r_0}, we have

$$\begin{aligned}
&\|x - y\|_{C(\mathbb{R}_+)} \\
&= \left| h(t, x(t)) \int_0^\infty k(t - \tau) f(\tau, x(\tau)) d\tau - h(t, y(t)) \int_0^\infty k(t - \tau) f(\tau, y(\tau)) d\tau \right| \\
&\leq \left| h(t, x(t)) \int_0^\infty k(t - \tau) f(\tau, x(\tau)) d\tau - h(t, x(t)) \int_0^\infty k(t - \tau) f(\tau, y(\tau)) d\tau \right| + \left| h(t, x(t)) \int_0^\infty k(t - \tau) f(\tau, y(\tau)) d\tau - h(t, y(t)) \int_0^\infty k(t - \tau) f(\tau, y(\tau)) d\tau \right| \\
&\leq \left[|h(t, x(t)) - h(t, 0)| + |h(t, 0)| \right] + |h(t, x(t)) - h(t, y(t))| \int_0^\infty |k(t - \tau) f(\tau, y(\tau))| d\tau \\
&\leq (a|x(t)| + b) \int_0^\infty k(t - \tau) |f(\tau, x(\tau)) - f(\tau, y(\tau))| \, d\tau + a|x(t) - y(t)| \times \int_0^\infty |k(t - \tau)| m(|y(\tau)|) d\tau \\
&\leq (a|x(t)| + b) Kl |x(t) - y(t)| + aKl_1 |x(t)| |x(t) - y(t)| \\
&\leq \left((a|x(t)| + b) Kl + aKl_1 |x(t)| \right) |x(t) - y(t)| \\
&\leq \left((a\|x\| + b) Kl + aKl_1 \|x\| \right) \|x - y\|_{C(\mathbb{R}_+)}.
\end{aligned}$$

Therefore we have.

$(1 - (a\|x\| + b)Kl + aKl_1\|x\|)\|x - y\|_{C(\mathbb{R}_+)} \leq 0.$

$(1 - (ar_0 + b)Kl + aKl_1 r_0)\|x - y\|_{C(\mathbb{R}_+)} \leq 0,$

which implies that $\|x - y\|_{C(\mathbb{R}_+)} \leq 0$, thus we have $\|x - y\|_{C(\mathbb{R}_+)} \to 0, \Rightarrow x = y$ this complete the proof.

In the following we will present an example to ensure the assumptions of our existence theorem.

5. Example

Consider the following quadratic integral equation,

$$x(t) = te^{-t} + \frac{1}{4} x(t) \int_0^\infty -(t - s) e^{-(t-s)^2} \arctan\left(\frac{x^2(s)}{1+s^2}\right) ds, \quad t \geq 0 \tag{5}$$

in our example we have $g(t) = te^{-t}$, $h(t, x(t)) = \frac{1}{4} x$, further notes that $k(t - s) = -(t - s)e^{-(t-s)^2}$ and $f(s, x) = \arctan\left(\frac{x^2}{1+s^2}\right)$.

first of all observe that $g(t)$ satisfies assumption (i) with norm $\|x\| = \frac{1}{e}$.

Further notes that the function $h(t, x)$ satisfies assumption (ii) with $a = \frac{1}{4}$ since $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and

$$|h(t, x) - h(t, y)| = \frac{1}{4} |x - y|.$$

for $x, y \in \mathbb{R}$ and $t \in \mathbb{R}_+$. It is easy to show that this function satisfies assumption(iii), indeed, taking an arbitrary nonnegative function $x \in C(\mathbb{R}_+)$ and $t, s \geq 0$ such that $t \geq s$, we obtain

$$|(Hx)(s) - hx(t)| - [(Hx)(s) - hx(t)] = |h(s, x(s)) - h(t, x(t))| - [h(s, x(s)) - h(t, x(t))]$$

$$= \left|\frac{1}{4}x(s) - \frac{1}{4}x(t)\right| - \left[\frac{1}{4}x(s) - \frac{1}{4}x(t)\right]$$

$$= \frac{1}{4}\left(\left|\frac{1}{4}x(s) - \frac{1}{4}x(t)\right| - \left[\frac{1}{4}x(s) - \frac{1}{4}x(t)\right]\right) \leq \frac{1}{4}d(x).$$

Hence, we have

$$d(Hx) \leq ad(x)$$

From assumption (iv) we have $K = \frac{1}{2}$.
Indeed

$$\int_0^\infty -(t-s)e^{-(t-s)^2}\, ds = \frac{-1}{2}e^{-(t-s)^2}\Big]_0^\infty = \frac{1}{2}e^{-t^2},$$

thus, $K = \frac{1}{2}$. Next, we note that the function $f(s, x)$ satisfies assumption (v). Moreover, we have
$|f(s, x)| \leq x^2$,
so the function $m(x) = x^2$.
Now, let us consider the inequality from the assumption (vi), which has the form

$$\|g\| + (ar_0 + b)Km(r_0) \leq r_0,$$

$$\frac{1}{e} + \left(\frac{1}{4}r_0\right)\frac{1}{4}r_0^2 \leq r_0 \qquad (6)$$

Using the standard methods of differential calculus we can verify that the function $y(r_0) = r_0 - \frac{1}{8}r_0^3$ attains its maximum at the point $r_0 = 2\sqrt{\frac{2}{3}}$, and $y(r_0) = \frac{4}{3}\sqrt{\frac{2}{3}} \geq \frac{1}{e}$. So the number r_0 is a positive solution of the inequality(6). finally taking in account all above established facts and the Theorem (3.1) we conclude that the above example has at least one solution defined, continuous and nondecreasing on $\mathbb{R}_+$.

Acknowledgement
The authors sincerely thank the anonymous reviewers and editor for their valuable suggestions, which improved the Presentation of this paper.

References

i. *Aghajani, J. Bana´s, Y. Jalilian, "Existence of solutions for a class of nonlinear Volterra singular integral equations", J. Comput. Math. Appl. 62(2011), 1215-1227. https://doi.org/10.1016/j.camwa.2011.03.049*

ii. *Aghajani, R. Allahyari, M. Mursaleen, "A generalization of Darbo's theorem with applicationto the solvability of systems of integral equations", J. Comput. Appl. Math. 260(2014), 68-77. https://doi.org/10.1016/j.cam.2013.09.039*

iii. *D. Beglov, B. Roux, "An integral equation to describe the solvation of polar molecules in liquidWater", J. Phys. Chem. B. 101(1997), 7821-7826. https://doi.org/10.1021/jp971083h*

iv. *G. Darbo, "Punti untiti in transformazioni a condominio noncompatto", Rend. Sem. Mat. Univ.Padora 24 (1955), 84-92.*

v. *H.W. Hethcote, D.W. Tudor, "Integral equation models for endemic infectious diseases",J.*
   a. *Math. Biology. 9(1980), 37-47.*

vi. *HA. Schenck, "Improved integral formulation for acoustic radiation problem", J A coust SodAm.44 (1968), 41-58. https://doi.org/10.1121/1.1911085*

vii. *I.M. Olaru, "Generalization of an integral equation related to some epidemic models",*
   i. *Carpathian J. Math. 26(1) (2010), 92-96.*

viii. J. Bana´s , A. Martinon, "Monotonic solutions of quadratic integral equation of voltera type",Comput. Math. Appl. 47(2004), 271-279. https://doi.org/10.1016/S0898-1221 (04)90024-7

ix. J. Bana´s , B. Rzepka, "Monotonic solutions of quadratic integral equation of fractional order",J. Math. Anal. Appl. 332(2007), 1370-1378.

x. J. Banas and K. Goebel, "Measures of Noncompactness in Banach Spaces", Lect. Notes in Math.60, M. Dekker, New York and Basel, 1980.

xi. J. Bana´s, M. Lecko and w. G. El-Sayed, "Existence theorem of some quadratic integral equations", J. Math. Anal. Appl. 222(1998), 276-285. https://doi.org/10.1006/jmaa.1998.5941

xii. J. Bana´s and M. Pasalawska-Poludnik, "Monotonic Solutions of Urysohn Integral equation onUnbounded interval", Comput. Math. Appl. 47(2004), 1947- 1954.

xiii. J. Banas and L. Olszowy, "Measures of noncompactness related to monotonicity", Comment. Math. 41 (2001), 13-23.

xiv. J. Bana´s and W.G. El-Sayed, "Solvability of functional and integral equations in some classes

i. of integrable functions", Politechnika Rzeszowska, Rzeszow, 1993.

xv. J. Appell, P.P. Zabrejko, "Nonlinear Superposition operator", Cambridge Tracts in Mathematics. 95, Cambridge University Press, 1990. https://doi.org/10.1017/CBO9780511897450

xvi. [J. Sladek, V. Sladek, S.N. Atluri, "Local boundary integral equation (LBIE) method for solving problem of elasticity with nonhomogeneous material properties", Computational Mechanics.

i. 24(2000), 456-462.

xvii. J. Bana´s, A. Chlebowicz, "On existence of integral solutions of a functional integral equation under Carath´eodory conditions", Nonlinear Anal, Vol. 70, No. 9, pp. 3172-3179, 2009.

xviii. M.A. Darwish, "On monotonic solutions of a singular quadratic integral equation with supremum", Dynam. Systems appl. 17(20008), 539-550.

xix. M.A. Darwish, " On quadratic integral equation of fractional orders",J. Math. Anal. Appl.

a. 311(2005), 112-119.

xx. M.A. Darwish, "On Erd´elyi -Kober fractional Urysohn-Volterra quadratic integral equation", Appl. Math. Comput. 273(2016), 562-569. https://doi.org/10.1016/j.amc.2015.10.040

xxi. M.A. Darwish, K. Sadarangani, "On quadratic integral equation with supremum involving

a. Erd´elyi -Kober fractional order", Mathematische Nachrichten. 288(5-6)(2015),

b. 566-576.

xxii. M. Benchohra, D. Seba, " Integral equations of fractional order with multiple time delays in

a. Banach spaces,Electron", J. differential Equations. 56(2012), 1-8.

xxiii. M. Cichon, M.A. Metwali, "On quadratic integral equations in Orlicz spaces", j. Math. Anal. Appl. 387(2012), 419-432. https://doi.org/10.1016/j.jmaa.2011.09.013

xxiv. M.M. El-Borai, W.G. El-Sayed, M.I. Abbas ,"Monotonic Solutions of a Class of Quadratic Singular Integral Equations of Volterra type", Int. J. Contemp. Math. Sci.2 (2007), 89-102. Int. J. Contemp. Math. Sci.2 (2007), 89-102.

xxv. R. Agarwal, M. Meehan, D. O'Regan, "Fixed point Theory and Applications",

a. Cambridge University Press, 2004.

xxvi. R.P. Agarwal, B. Samet, "An existence result for a class of nonlinear integral equations of fractional orders", Nonlinear Analysis. Modelling and Control. 21(5)(2016), 616- 629.

a. https://doi.org/10.15388/NA.2016.5.10

xxvii. R. Hilfer, "Applications of Fractional Calculus in Physics", World Scientific. Singapore, 2000.

a. https://doi.org/10.1142/3779

xxviii. S. Chandraseker, "Relative transfer", Dover publications, New York, 1960.

xxix. S.Hu, M. Khavani and W. Zhuang, " Integral equations arising in the kinetic theory of gases",

a. Appl. Anal. 34(1989), 261-266. https://doi.org/10.1080/00036818908839899

xxx. U. C¸akan, I. Ozdemir, "An application of Darbo fixed-point theorem to a class of functional integral equations", Numer. Funct. Anal. Optim. 36(2015), 29-40.

a. https://doi.org/10.1080/01630563.2014.951771

xxxi. *W.G. El-Sayed, A.A.H. Abd El-Mowla, "Nonincreasing Solutions for Quadratic Integral Equations of Convolution Type", J. Math. App (JMA). No 42, pp 95-107 (2019)*

xxxii. *W.G. El-Sayed, B. Rzepka, "Nondecreasing solutions of a quadratic integral equation of Urysohn type", Comput. Math. Appl. 51 (2006), 1065-1074.*

a. *https://doi.org/10.1016/j.camwa.2005.08.033*

xxxiii. *W. Al Sayed, M.A. Darwish, "On the Existence of Solutions of a Perturbed Functional Integral Equation in the Space of Lebesgue Integrable Functions on R+", J. Math . App(JMA). No.41, pp.19-27, 2018.*